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Algebraic aspects of almost complex manifolds and special metrics

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based on joint with
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Manifolds of interest:

$$\text{Almost Hermitian} \mapsto \text{Almost Complex} \mapsto \text{Smooth} \mapsto \text{Topological}$$

Questions:

1. What do various types of special metrics tell us about the underlying compatible almost complex structure?
2. What do various types of almost complex structures tell us about the underlying smooth structure or topology?
3. In each case, we may study admissibility, or obstructions.

Rather than seeking a classification, we're looking for *organizing principles* and *interesting structures* through which to **increase our understanding**.

The ideas are successful when they *inform*, and *are informed by*, other areas of mathematics.

- M smooth manifold of dimension $2n$.
- Almost complex structure: $J : TM \longrightarrow TM$; $J^2 = -\mathrm{Id}$.

  Examples: (M, J) complex manifold (i.e. holomorphic atlas). In general, a choice of J is a section of a bundle:

$$\begin{tikzcd}\{\text{linear complex structures on } T_pM\} \arrow[r] & \mathcal{J} \arrow[d, shift right] \\ & M \arrow[u, bend right]\end{tikzcd}$$

  Note that the fiber is non-compact: e.g. $J_t = \begin{bmatrix} 0 & t \\ -1/t & 0 \end{bmatrix}$.

  For $(n = 1)$, the fiber is $\mathbb{R}^2 - \{\text{line}\} \simeq S^0$.

  In general, the fiber is $Gl(2n, \mathbb{R})/Gl(n, \mathbb{C})$ of dimension $n(2n) = 2n^2$.

- Remark: Existence depends on TM (the smooth structure!).

Let's look in low dimensions:

($2n = 2$): The "twistor bundle" $\mathcal{J} \to M$ is the orientation double cover (up to homotopy), since

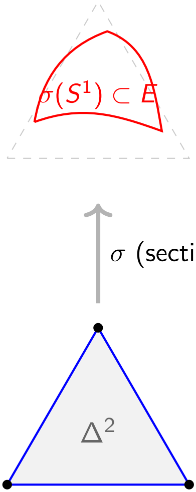
$$\begin{tikzcd} S^0 \simeq (\mathbb{R}^2 - \{line\}) \arrow[r] & \mathcal{J} \arrow[d, shift right] \\ & M \arrow[u, shift right, bend right] \end{tikzcd}$$

So, M has an almost complex structure iff M is orientable, i.e. the first Stiefel-Whitney class $w_1 \in H^1(M; \mathbb{Z}_2)$ vanishes.

Exercise: Orientability is a necessary condition in all dimensions. Must be careful about "compatibility with a preferred orientation".

Why are there cohomological obstructions?

Obstruction Theory



Obstruction Mechanics:

1. Define σ on $(n-1)$-skeleton.
2. Boundary of n-simplex is a sphere:

$$\partial\Delta^n \cong S^{n-1}$$

3. $\sigma|_{\partial\Delta^n}$ defines a map $S^{n-1} \to F$, yielding an element in $\pi_{n-1}(F)$.
4. Evaluating on all n-simplices yields a cocycle c^n; note $\partial(\partial\Delta) = 0$.

First obstruction is in $H^{k+1}(M; \pi_k(F))$, for first non-zero homotopy group $\pi_k(F)$ of fiber F.

Theorem (Wu, 1952)

A smooth, compact, connected, oriented 4-manifold M admits an almost complex structure if and only if there exists a cohomology class $c \in H^2(M;\mathbb{Z})$ such that:

1. *The class c reduces mod 2 to the second Stiefel-Whitney class of M:*

$$c \equiv w_2(M) \pmod 2$$

2. *The signature theorem relation holds:*

$$c^2[M] = 2\chi + 3\sigma,$$

$\chi(M)$ is Euler characteristic; $\sigma(M)$ is the signature.

If such an almost complex structure exists, the class c corresponds to its first Chern class $c_1(TM)$.
Example: S^4 is not almost complex $(0 \neq 2 \cdot 2 + 3 \cdot 0)$.

In dimension six, assuming simply-connectedness, the first obstruction to an (oriented) almost complex structure is in

$$H^3\left(M^6; \pi_2(SO(6)/U(3))\right)$$

Fact: $SO(6)/U(3) \cong \mathbb{C}P^3$, therefore the higher obstructions in $H^{k+1}(M^6; \pi_k(\mathbb{C}P^3))$ vanish for $3 \leq k \leq 6$.

So: *In dimension 6, it's easy to be almost complex*.

Example: $S^2 \times S^4$ is almost complex (although S^4 is not). Also, rational 6-spheres: infinitely many topologically distinct.

Remark: K-theory, spheres, products of even dimensional spheres that are almost complex are products of S^2, S^6, and $S^2 \times S^4$.

1. In 1961 Massey explained the first obstructions in all dimensions: the "fractional part of integral Stiefel-Whitney classes" or "fractional part of certain non-linear combinations of characteristic classes on lower skeleta".
2. For the state of the art, see: "Obstructions to almost complex structures follow Massey," by Albanese and Milivojevic, 2024 preprint. Includes a new expression for the second obstruction to a complex structure on any rank 6 bundle, in terms of characteristic classes.

   Open Problem: Can all obstructions be explained in terms of characteristic classes?

3. It is likely that differential cohomology theories play an important role in understanding the obstructions that are realized as "fractional parts". Differential cohomology $\hat{H}$ incorporates the Bockstein homomorphism defining integral Steifel-Whitney classes. Differential K-theory $\hat{K}$ incorporates geometric representatives for cohomology classes, relates them in certain exact sequences involving differential forms, and refines the A-S index theory used in study complex surfaces.

Assuming the existence of an almost complex structure: what are the tools available for studying these? And what organizational concepts do they lend us?

Keeping in mind the tension between *computability* and *information loss*, we first select from the highly computable tools from topology:

Differential forms, $(\Omega^*(M), d)$, which know the smooth structure.

de Rham's Theorem: $H^*(M; \mathbb{R}) \cong \frac{Ker\, d}{Im\, d}$. (cohomology mod torsion).

1. Highly computable
2. Many inequivalent structures have same invariants, i.e. there is no nice dictionary between maps of space and maps of cohomology (Exercise!)
3. Rational homotopy theory resolves this: don't take cohomology!

Sullivan's insight: for any nice space, one can construct from the (rational) differential forms the so-called *minimal model*:

a free commutative differential graded algebra (cdga) ΛV equipped with a differential d and map

$$\Lambda V \to \Omega^*(M),$$

yielding an isomorphism on cohomology. Unique up to isomorphism.

Theorem (Sullivan)

1. $\pi_k(M) \otimes \mathbb{Q} = (V^k)^*$, *i.e. one can compute rational homotopy groups from the minimal model by counting generators in each degree.*
2. *Maps of spaces (up to rational homotopy) are in bijection with homotopy classes of maps of minimal models.*

Modulo torsion, this resolves the tension between computability and loss of information.

Why not try for more?

1. (Gompf, 1988) Every finitely presented group is π_1 of a compact complex manifold of complex dimension 3.
2. Fundamental groups are relatively difficult to compute. (But $\pi_1 \otimes \mathbb{Q}$ is computable.)
3. (Le Brun, 1998) Chern numbers of a compact complex 3-folds are not topologically determined: can vary the complex structure on same underlying smooth manifold and make Chern numbers unbounded.
4. Thus, "topological invariants" of (almost) complex manifolds may neglect "analytic information", but there is additional "analytic information" in induced *elements* of the groups, algebras, etc. which should be carried along.

The de Rham complex of an almost complex manifold has:

$$\Omega^k(M) \otimes \mathbb{C} = \bigoplus_{p+q=k} \Omega^{p,q}(M)$$

where $\Omega^{p,q}(M)$ denotes smooth sections of the bundle $\Lambda^p(T^{*\,1,0}M) \otimes \Lambda^q(T^{*\,0,1}M)$ (i.e., forms of type (p,q)).

Here, J induces a pointwise decomposition of the complexified cotangent bundle into $+i$ and $-i$ eigenspaces ($T^{*,1,0}M$ and $T^{*,0,1}M$).

Exercise: The splitting of T^*M into conjugate subspaces in this way completely determines J.

There's an action of J induced on $\Omega^*(M)$, by extending J as a derivation given by

$$J_{der} = i(p-q) \qquad \text{on} \qquad \Omega^{p,q}(M).$$

Claim:

$$e^{\frac{\pi}{2} J_{der}} = J_{alg}.$$

where J_{alg} is the extension of J as an algebra automorphism.
Proof:

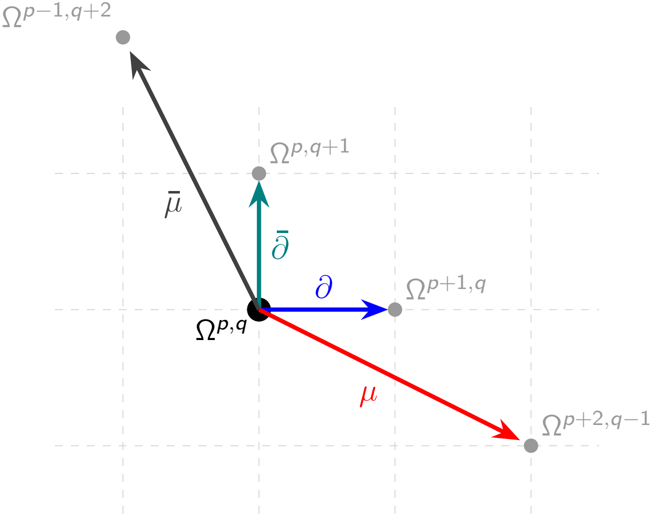
$$e^{\frac{\pi}{2} J_{der}} = e^{\frac{\pi}{2} i(p-q)} = i^{p-q} = J_{alg}.$$

Generalizes Euler's formula $e^{\pi i} + 1 = 0$ on the almost complex manifold $\mathbb{C}$, i.e. $e^{\frac{\pi}{2} i} = \sqrt{-1}$)

Claim: (Ω, d) is a "quad-complex", i.e.

$$d = \mu + \partial + \bar{\partial} + \bar{\mu}$$

Proof: graded components of a derivation are determined by action on a set of generators; so it suffices to check on functions and 1-forms.



(Integrability Exercise) The following are equivalent:

1. $\bar{\mu} = 0$.
2. $d = \partial + \bar{\partial}$, i.e. $(\Omega, d = \partial + \bar{\partial})$ is a bicomplex.
3. The ideal $\Omega^{\geq 1}$ of Ω^* is a differential ideal.
4. $$[d, J_{der}] = J_{alg}^{-1} d J_{alg}$$

   Compatibility of ad and Ad (reminiscient of Kähler identities).

(Newlander-Nirenburg) Any of these equivalent conditions are equivalent to the existence of a holomorphic atlas.

Last condition implies $d \equiv 0$ on H_{d^c}, so that "d^c diagrams" are defined for all complex manifolds:

Allows us to use techniques from rational homotopy theory to study obstructions to existence of certain complex structures (third talk in series).

Structure Theory?

What are the organizing principles here?

Common technique in mathematics is to decompose complicated structures into “irreducible components”, i.e. a *structure theory*.

Ex: (Ω, J_{der}) is a direct sum of components on which J_{der} acts by scalar multiplication.

Open problem: develop a structure theory of “quad-complexes”, for which every quad-complex is a direct sum of irreducible quad-complexes.
(Solution for bicomplexes: second talk by Jonas Stelzig).

A second example of “decomposing” is the Grey-Hervella classification of almost Hermitian metrics.

Question: Is there a single organizing framework that presents almost-Hermitian manifolds with their metric and the de Rham complex into a direct sum of irreducibles?

In spite of not having ideal answers to these questions, we can still make some progress for general almost-complex and almost Hermitian manifolds.

The following short story recovers fundamental facts about compact complex surfaces, generalizing it to compact almost complex manifolds. As a corollary, in the integrable case, we get the well-known:

1. (*Degeneration*) $b^k = \sum_{p+q=k} h^{p,q}$.
2. (*Signature*) $\sigma = 4\chi - e = \sum (-1)^q h^{p,q}$.
3. (*Noether's formula*) $12\chi = c_1^2 + e$.

Here b^k is k-th Betti number, $\sigma := b^+ - b^-$ is signature; c_1 is the first Chern class, $\chi := \sum (-1)^q h^{0,q} = 1 - \mathfrak{q} + p_g$ is holomorphic Euler characteristic, e is Euler characteristic.

So, Hodge numbers $h^{p,q}$ are determined by Betti numbers and signature, i.e. depend only on the oriented real cohomology ring.

Recall the exterior derivative d decomposes:

$$d = \bar{\mu} + \bar{\partial} + \partial + \mu$$

where $\bar{\mu} : \Omega^{p,q} \to \Omega^{p-1,q+2}$ and $\mu : \Omega^{p,q} \to \Omega^{p+2,q-1}$.

The $d^2 = 0$ Relations

The identity $d^2 = 0$ yields several equations by bidegree. Critically:

- $\bar{\mu}^2 = 0$
- $\bar{\mu}\bar{\partial} + \bar{\partial}\bar{\mu} = 0$
- $\bar{\partial}^2 + \bar{\mu}\partial + \partial\bar{\mu} = 0$

Consequence: $\bar{\partial}^2 \neq 0$ unless J is integrable ($\bar{\mu} = 0$).

If J is non-integrable, $\bar{\partial}$ does not define a cohomology in the standard sense.

- We can still define the $\bar{\partial}$-harmonic forms $\mathcal{H}_{\bar{\partial}}^{p,q} = \mathrm{Ker}(\Delta_{\bar{\partial}})$.
- However, $\dim \mathcal{H}_{\bar{\partial}}^{p,q}$ is **metric-dependent**.
- **Goal:** Find a metric-independent, finite-dimensional analog of Hodge numbers.

The standard Hodge filtration $F^p_{std} = \bigoplus_{i \geq p} \Omega^{i,n-i}$ is not stable under d because of $\bar{\mu}$.

We define the **modified filtration**:

$$F^p\Omega^n := (\operatorname{Ker} \bar{\mu} \cap \Omega^{p,n-p}) \oplus \bigoplus_{r>p} \Omega^{r,n-r}$$

Proposition (Cirici, W.)

This filtration satisfies $d(F^p\Omega^n) \subseteq F^p\Omega^{n+1}$. The resulting spectral sequence (E_r, d_r) converges to $H^*_d(M, \mathbb{C})$.

- **The E_1 Page:** $H^{p,q}_{\bar{\mu}} = \frac{Ker\,\bar{\mu} \cap \Omega^{p,q}}{Im\,\bar{\mu} \cap \Omega^{p,q}}$
- **The E_2 Page:** $H^{p,q}_{\bar{\partial}}(H^{p,q}_{\bar{\mu}}))$.

We **define** $h^{p,q} := \dim E_2^{p,q}$. Agrees with integrable case.

Theorem 1 (Degeneration)

For any compact almost complex 4-manifold, the spectral sequence degenerates at the second page: $E_2 = E_\infty$.

Corollary: $b^k = \sum_{p+q=k} h^{p,q}$ and $e = \sum_{p+q=k} (-1)^{p+q} h^{p,q}$.

Theorem 2 (Duality)

For compact almost complex 4-manifolds:

$$h^{p,q} = h^{2-p,2-q}$$

Technical Sketch.

1. Identify $E_1^{p,q}$ with a space of $\bar{\partial}$-harmonic forms in $\mathrm{Ker}(\bar{\mu})$.
2. Show the Hodge star $\bar{\star}$ maps $\mathrm{Ker}(\bar{\mu}) \to \mathrm{Ker}(\mu)$.
3. Establish a pairing between $E_2^{p,q}$ and $E_2^{2-p,2-q}$ using the L^2-inner product. $\square$

Define

$$\chi = h^{0,0} - h^{0,1} + h^{0,2} \qquad \mathfrak{q} = h^{0,1} \quad \text{the } \textit{irregularity}.$$

Theorem

For any almost complex 4-manifold:

$$\widetilde{\sigma} = 4\chi - e$$

where

$$\widetilde{\sigma} := \sum_{p,q=0}^{2} (-1)^q h^{p,q}$$

Remark: the signature theorem and Todd class formulas

$$\sigma = \frac{1}{3}p_1 = \frac{1}{3}(c_1^2 - 2e) \qquad Td = \frac{1}{12}(c_1^2 + e).$$

imply

$$\sigma - \widetilde{\sigma} = 4(Td - \chi) \equiv 0 (\mathrm{mod}\, 4),$$

So $\sigma = \widetilde{\sigma}$ for a complex surface. But this does not hold in general for almost complex manifolds

In the non-integrable case, one can show

$$h^{2,0} = 0 \qquad \text{and} \qquad h^{1,1} = b^2,$$

Proof uses unique continuation theorem for harmonic forms.

Theorem (Cirici, W.)

For compact non-integrable almost complex manifolds:

1. *All $h^{p,q}$ determined by the topology together with the irregularity $\mathfrak{q} := h^{0,1}$,*
2. *$\mathfrak{q}$ depends on the almost complex structure, BUT*
3. *Topological bounds: $b^1 \leq 2\mathfrak{q} \leq 2b^1$,*
4. *$\mathfrak{q}$ is lower semi-continuous under small deformations of the underlying almost complex structure.*

$$KT := H_{\mathbb{Z}} \times \mathbb{Z} \backslash H \times \mathbb{R}$$

Betti numbers: $b^1 = 3, b^2 = 4$.

In this case we have $\mathfrak{q} \in \{2, 3\}$, which give the Hodge-de Rham diamonds

$$\begin{array}{ccccc} & & 1 & & \\ & 1 & & 2 & \\ 1 & & 2 & & 1 \\ & 2 & & 1 & \\ & & 1 & & \end{array} \qquad \begin{array}{ccccc} & & 1 & & \\ & 1 & & 2 & \\ 0 & & 4 & & 0 \\ & 2 & & 1 & \\ & & 1 & & \end{array} \qquad \begin{array}{ccccc} & & 1 & & \\ & 0 & & 3 & \\ 0 & & 4 & & 0 \\ & 3 & & 0 & \\ & & 1 & & \end{array}$$

Type 1   Type 2   Type 3

and the associated invariants

	$\mathfrak{q}$	$\widetilde{\sigma}$	σ	χ
Type 1 (integrable)	2	0	0	0
Type 2 (non-integrable)	2	-4	0	-1
Type 3 (non-integrable)	3	-8	0	-2

Filiform Nilmanifold

Let $\mathfrak{g}$ be the Lie algebra with brackets $[X_1, X_j] = X_{j+1}$ for $j = 2, 3$.

- $b^1 = 2, b^2 = 2$.
- **Theorem:** $M = \Gamma / G$ admits no integrable structure. (sketch)
- **Hodge Data:** Topology implies $\mathfrak{q} \in \{1, 2\}$. If $\mathfrak{q} = 2$, the diamond has $h^{1,1} = 2$, but $h^{0,1} = 2$.

$$\begin{array}{ccccc} & & 1 & & \\ & 1 & & 1 & \\ 0 & & 2 & & 0 \\ & 1 & & 1 & \\ & & 1 & & \end{array}$$

Type 1

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 2 & \\ 0 & & 2 & & 0 \\ & 2 & & 0 & \\ & & 1 & & \end{array}$$

Type 2

	$\mathfrak{q}$	$\widetilde{\sigma}$	σ	χ
Type 1 (non-integrable)	1	0	0	0
Type 2 (non-integrable)	2	4	0	-1

Consider for $t > 0$:

$$J_t = \begin{bmatrix} 1 & -2\operatorname{csch} t & 0 & 0 \\ \sinh t & -1 & 0 & 0 \\ 0 & 0 & -1-\sqrt{2} & -2(2+\sqrt{2})\operatorname{csch} t \\ 0 & 0 & \sinh t & 1+\sqrt{2} \end{bmatrix}.$$

where $\sinh t = (e^t - e^{-t})/2$ and $\operatorname{csch} t = 1/\sinh t$.

Note that $\lim\limits_{t\to\infty} J_t$ does not exist:

$$\lim_{t\to\infty} J_t = \begin{bmatrix} 1 & 0 & 0 & 0 \\ +\infty & -1 & 0 & 0 \\ 0 & 0 & -1-\sqrt{2} & 0 \\ 0 & 0 & +\infty & 1+\sqrt{2} \end{bmatrix}.$$

- One can show: $\bar{\mu}_{J_t} \to 0$ as $t \to \infty$, as $\bar{\mu}$ is the dual of the Nijenhuis tensor N, and it is completely determined by (Ignore this!)

$$
\begin{aligned}
N_t(X_1, X_2) &= 0 \\
N_t(X_1, X_3) &= (4 + 4\sqrt{2})\,\mathrm{csch}\,t\;X_3 \\
N_t(X_1, X_4) &= 2(2 + \sqrt{2})\,\mathrm{csch}\,t\,[2(2 + \sqrt{2})\,\mathrm{csch}\,t\;X_3 - \sqrt{2}\;X_4] \\
N_t(X_2, X_3) &= 4\,\mathrm{csch}\,t\,[(2 + \sqrt{2})\,\mathrm{csch}\,t\;X_3 - (1 + \sqrt{2})\;X_4] \\
N_t(X_2, X_4) &= -4(2 + \sqrt{2})\,\mathrm{csch}^2\,t\;X_4 \\
N_t(X_3, X_4) &= 0.
\end{aligned}
$$

  So $N_t \to 0$ uniformly on $\Gamma\backslash G$ as $t \to \infty$.
- So, any such filiform manifold $\Gamma\backslash G$ has no complex structure, but has a family J_t with arbitrarily small Nuijenhuis tensor.
- We have no conceptual explanation for this!

Open Problem on $N_t \to 0$

Many more examples like this exist on 4 and 6 dimensional nil and solvmanifolds (Fernandez, Shin, W.). In particular,

- (Goze-Remm, 2002): Any 6-dimensional filiform manifold (with brackets $[X_1, X_j] = X_{j+1}$ for $j = 2, 3, 4, 5$) has no left invariant complex structure.
  Yet, there is a family J_t with $N_t \to 0$.
  (Terribly unsatisfying construction/proof!)
- (Fernández-Gray, 1990) A compact 4-dimensional parallelizable manifold with $b_1 = 2$ has no complex structure. There is such a solvmanifold M^4 which satisfies all known cohomological properties of Kähler manifolds and is formal but is not complex. This manifold has family J_t with $N_t \to 0$.

Does every compact nil or solv-manifold have a (left invariant) family of almost complex structures J_t with $N_t \to 0$? WHY?
Resolving this question might yield some insight on whether the analogous question is true for all manifolds.

Recall M is almost complex iff M is almost-Hermitian iff M is almost symplectic.

Reason: fibers $Gl(2n, \mathbb{R})/Gl(n, \mathbb{C}) \simeq Gl(2n, \mathbb{R})/Sp(2n, \mathbb{R})$.

What is the algebraic structure presented by an almost Hermitian or almost symplectic manifold?

Remark: not even known without the additional structure.

Will explain some progress uncovering a homotopy-theoretic structure related to the Kähler identities.

Let (M, ω, J) be a compact Kähler manifold. The standard Kähler identity states:

$$[\Lambda, d] = \pm d^c,$$

where $\Lambda = L^*$ is the interior product with the symplectic form ω, and $d^c = J_{alg}^{-1} d J_{alg}$.

The Conjugation Formula

Computing $Ad_{e^{t\Lambda}}(d)$:

$$e^{-t\Lambda} d e^{t\Lambda} = \sum_{k=0}^{\infty} \frac{t^k}{k!} \mathrm{ad}_\Lambda^k(d) = d + t[\Lambda, d] + \frac{t^2}{2}[\Lambda, [\Lambda, d]] + \dots$$

In the classical Kähler case, infinite sum collapses to two terms:

$$e^{-t\Lambda} d e^{t\Lambda} = d \pm t d^c$$

with $[\Lambda, d^c] = 0$ and the higher commutators vanishing.

Theorem

(Almost Kähler identities, Tomassini-Wang, 2021)
If (M, ω, J) *is an almost symplectic manifold then*

$$[\Lambda, d] = d^{\omega} + [\Lambda, [d, L]]^{\omega}$$

where the superscript is "symplectic adjoint" i.e. conjugation by symplectic-star $\star_{\omega}$ *(up to sign).*

Exercise:

$$[\Lambda, [\Lambda, d]] = [\Lambda, d^{\omega} + [\Lambda, [d, L]]^{\omega}] = -2[d, L]^{\omega}$$

and

$$[\Lambda, [\Lambda, [\Lambda, d]]] = 0$$

The Almost Kähler Expansion

Computing $Ad_{e^{t\Lambda}}(d)$:

$$e^{-t\Lambda} d e^{t\Lambda} = d + t(d^\omega + [\Lambda, [d, L]]^\omega)) - t^2[d, L]^\omega$$

Note:

Operator	d	$d^\omega + [\Lambda, [d, L]]^\omega$	$[d, L]^\omega$
degree	1	-1	-3
alg. order	1	2	3

This quadratic termination is a special case of a higher homotopy structure.

Definition

A commutative BV_∞-algebra is a commutative, associative graded algebra A equipped with a sequence of linear operators $\Delta_k : A \to A$ of degree $1 - 2k$ ($k \geq 0$) satisfying:

$$\sum_{i+j=n} [\Delta_i, \Delta_j] = 0$$

where each Δ_k acts as a differential operator of algebraic order $\leq k + 1$ relative to the product.
The above equation is equivalent to

$$(d + \Delta_1 + \cdots + \Delta_n)^2 = 0$$

for all n.

Since Λ is degree -2 and algebraic order 2, the components of $Ad_{e^{t\Lambda}}(d)$ define a commutative BV_∞-algebra.

Unlike strict BV algebras $(d, \Delta 0)$, BV_∞ structures are robust under homotopy equivalences of underlying chain complexes.

- There is a good notion of quasi-isomorphism of BV_∞ structures.
- Transfer Theorem: Given a BV_∞ algebra A and a chain homotopy equivalence to a vector space H (such as the cohomology), the there is a quasi-isomorphic BV_∞ algebra on H.
- Transferred structure can be computed explicitly via "sum over tree" formulas.

In particular, the cohomology of an almost symplectic manifold has the structure of a commutative BV_∞ algebra, with operators Δ_k on cohomology, possibly all non-zero.

Theorem (Getzler, 1994)

The homology of the Deligne–Mumford moduli space of curves $H_*(\overline{\mathcal{M}}_{0,n+1})$ possesses an operadic structure that acts naturally on any Batalin–Vilkovisky (BV) algebra.

Such a structure is a called a hypercommutative algebra, and posses $m_n : A^{\otimes n} \to A$ with higher associativity conditions.

Theorem of Dotsenko, Shadrin, Vallette: at chain level, this commutative BV_∞-algebra defines an action of $\Omega H^{*+1}(\mathcal{M}_{0,*+1}) \simeq H^*(\overline{\mathcal{M}_{0,*+1}})$, yielding a "homotopy hypercommutative algebra".

Altogether, the cohomology of an almost symplectic manifold has an action of "chains on compactified moduli space of genus zero curves". These yield computable higher operations in cohomology that are non-trivial, even in the case of nilmanifolds (e.g. Kodaira-Thurston).

Outlook:

- The 3-term termination formula relies purely on the compatibility of d with the fundamental 2-form ω, meaning it belongs fundamentally to almost symplectic topology.
- This generalizes several discussions (symplectic, Poisson, Jacobi) and has real-Lagrangian version (omitted here).
- In the Kähler case, known to be "formal" (Cirici-Horel).
- Not much is known about the higher operations for other special metrics here.
- Reasonable to wonder if these BV_∞ operations and the induced moduli space actions (defined on the de Rham complex) bear any relationship to the higher quantum products or A-infinity structures found in Fukaya categories and Gromov-Witten theory.

Summary: There is still very much for us to pursue in the realm of "algebraic aspects of almost complex manifolds".

Thank you!